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Korovkin Closures for Positive Linear Operators

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INTRODUCTION

Let E and F be topological vector lattices, and let H denote a linear subspace of E. The Korovkin closure (or shadow) $\overline{H}_S^{\mathscr{P}}$ of H with respect to some linear lattice homomorphism S: $E \to F$ is the set of all $x \in E$ satisfying the following condition:

For each net $(T_i)_{i\in I}$ of positive linear operators, $(T_i(x))_{i\in I}$ converges to $S(x)$ provided that $\lim_{i\in I} T_i(y) = S(y)$ for all $y \in H$.

There are many sufficient conditions for an element $x \in E$ to belong to the Korovkin closure $\overline{H}_S^{\mathscr{P}}$ (see, e.g., [1], [3], [4], [8], [12], [13], [14], [16]). Although necessary conditions have been given for some particular vector lattices [1], [2], [3], [4], [11]) exact descriptions of Korovkin closures are rare the literature. It is the main intention of this note to fill this gap together with a parallel publication (see [5]). In fact, we characterize $\overline{H}_S^{\mathscr{P}}$ for an arbitrary topological vector lattice F.

In view of the applications, the operators T_i are often defined only on some linear subspace E_0 of E (e.g., $E = F = \mathscr{C}([0, 1]$, S is the identity operator and E_0 is the space of all polynomials on [0, 1]). Our characterization also carries over to this case. Replacing the nets of positive linear operators by sequences we obtain sequential Korovkin closures. Under some rather general assumptions sequential Korovkin closures are also characterized.

NOTATIONS AND DEFINITIONS

Throughout this note E will be a *vector lattice* and E_0, H will denote *linear subspaces of* E such that $H \subset E_0$. Furthermore, let F be a *Hausdorff topological vector lattice*[1] and let $S : E \to F$ be a linear lattice homomorphism. For each $x \in E$, we set

[1] See [7] for a definition of topological vector lattices.

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$$\begin{aligned}
H^x &:= \{y \in H: y \leqslant x\},\\
H_x &:= \{y \in H: y \geqslant x\},\\
\underset{\smile}{H}^x &:= \{\sup A: \varnothing \neq A \subset H^x, A \textit{ finite}\},\\
\hat{H}_x &:= \{\inf A: \varnothing \neq A \subset H_x, A \textit{ finite}\}.
\end{aligned}$$

Note that $\hat{H}_x$ and $\underset{\smile}{H}^x$ are downward and upward directed, respectively. An element $x \in E_0$ is called *(H, S)-affine* iff $H_x \neq \varnothing$, $H^x \neq \varnothing$ and $\lim_{y \in \hat{H}_x} S(y) = S(x) = \lim_{y \in \underset{\smile}{H}^x} S(y)$. The subset of all (H, S)-affine elements in E_0 will be denoted by $\mathscr{A}_S(H)$. Given a class $\mathscr{T}$ of linear operators from E_0 into F the *Korovkin closure* or *shadow* $\overline{H}_S^{\mathscr{T}}$ of H with respect to $\mathscr{T}$ and S is the set of all $x \in E_0$ such that $(T_i) \in \mathscr{T}$ and $\lim_{i \in I} T_i(y) = S(y)$ for all $y \in H$ implies $\lim_{i \in I} T_i(x) = S(x)$. In this note only subclasses of the class $\mathscr{P}$ of all nets of positive linear operators from E_0 into F will be considered for $\mathscr{T}$.

The proof of the following lower estimate for $\overline{H}_S^{\mathscr{P}}$ is only a slight modification of well-known arguments which have been repeatedly used by many authors ([1], [3], [4], [5], [13]). Hence we omit the proof.

THEOREM 1. *Each (H, S)-affine element of E_0 is contained in $\overline{H}_S^{\mathscr{P}}$.*

Before we prove the converse of Theorem 1 we give some examples

(1) (see [1], [2]) Let X be a topological space and let $F \subset \mathscr{C}(X)$ be a Hausdorff topological vector lattice of real-valued functions on X. For a linear subspace H of $\mathscr{C}(X)$ we define

$$H_0 := \{f \in \mathscr{C}(X): H_f \neq \varnothing \quad \text{and} \quad H^f \neq \varnothing\}.$$

If $S: \mathscr{C}(X) \to F$ is the natural embedding and $(T_i)_{i \in I}$ is a net of positive linear operators from H_0 into F converging to S on H, then $\lim_{i \in I} T_i(f) = f$ for all (H, S)-affine functions f in H_0. Thus we obtain Bauer's results if we choose $F = \{f \in \mathbb{R}^X: f \textit{ is bounded on each compact subset of } X\}$ endowed with the topology of pointwise resp. locally uniform convergence, or if we set $F = \mathscr{C}(X)$ endowed with the order topology.

(2) (see [3],) Let μ be a Borel measure on a locally compact space X. If F denotes the vector lattice of all equivalence classes of μ-a.e. finite, measurable functions on X with values in $\mathbb{R} \cup \{+\infty, -\infty\}$ (the equivalence relation being μ-a.e.-equality), then the topology of convergence in measure is compatible with the linear structure of F and makes F a topological vector lattice, which is not locally convex, in general. For $f \in \mathscr{C}(X)$ let $S(f) \in F$ be the equivalence class of f in F. Then $S: \mathscr{C}(X) \to F$ is a linear lattice homomorphism. Thus, by Theorem 1, if (T_i) is a net of positive linear maps from $\mathscr{C}(X)$ into F satisfying $\lim_{i \in I} T_i(h) = S(h)$ for all functions h of a subspace H of $\mathscr{C}(X)$ we have $\lim_{i \in I} T_i(f) = S(f)$ for each (H, S)-affine function f in $\mathscr{C}(X)$.

It will be shown in a forthcoming publication that (H, S)-affine elements are easy to describe in many cases of practical interest. Here, however, we shall go straight ahead to prove the converse of Theorem 1. The following lemma is crucial for understanding the rest of this note:

LEMMA 1. *Let V be an order-complete vector lattice. If $Q: H \to V$ is a positive linear mapping and if $\mathscr{F}_Q$ denotes the set of all positive linear extensions of Q on $H_0 := \{x \in E_0 : H_x \neq \varnothing \neq H^x\}$. Then*

$$\{L(f): L \in \mathscr{F}_Q\} = \{v \in V: \sup Q(H^f) \leqslant v \leqslant \inf Q(H_f)\} \qquad \text{for all} \quad f \in H_0 .$$

Proof. The mapping $\tilde{Q}: H_0 \to V$ defined by $\tilde{Q}(f) = \inf Q(H_f)$ is sublinear and $-\tilde{Q}(-f) = \sup Q(H^f)$ for each $f \in H_0$. Hence, if $L \in \mathscr{F}_Q$ and $f \in H_0$ we obtain

$$-\tilde{Q}(-f) = \sup L(H^f) \leqslant L(f) \leqslant \inf L(H_f) = \tilde{Q}(f).$$

Conversely, let $f \in H_0$ and suppose that $v \in V$ satisfies $-\tilde{Q}(-f) \leqslant v \leqslant \tilde{Q}(f)$. On the one-dimensional subspace $\mathbb{R}f \subset H_0$ generated by f we define a linear operator $T: \mathbb{R}f \to V$ by setting $T(\lambda f) = \lambda v$ for all $\lambda \in \mathbb{R}$. From the inequality $v \leqslant \tilde{Q}(f)$ (resp. $-v \leqslant \tilde{Q}(-f)$) it follows that $T(\lambda f) \leqslant \lambda \tilde{Q}(f) = \tilde{Q}(\lambda f)$ for all $\lambda \in \mathbb{R}_+$ (resp. $T(\lambda f) \leqslant -\lambda \tilde{Q}(-f) = \tilde{Q}(\lambda f)$ for all $\lambda \in \mathbb{R}_+$). Hence $T(\lambda f) \leqslant \tilde{Q}(\lambda f)$ for all $\lambda \in \mathbb{R}$. By the Hahn-Banach theorem for linear mappings into order-complete vector lattices there is a linear extension $L: H_0 \to V$ of T satisfying $L(g) \leqslant \tilde{Q}(g)$ for all $g \in H_0$. If $g \in H_0$, $g \leqslant 0$, we deduce $0 \geqslant \tilde{Q}(g) \geqslant L(g)$, hence L is positive. Moreover, since $\tilde{Q}(h) = -\tilde{Q}(-h)$ for all $h \in H$, we must have $L \in \mathscr{F}_Q$ which completes the proof.

For completeness let us first mention the elementary converses of Theorem 1, when E or F are order-complete (see [13]).

PROPOSITION 1. *If F is order-complete and the topology of F is order-continuous, or if E is an order-complete topological vector lattice with order-continuous topology and $S: E \to F$ is continuous, then the following statements are equivalent provided that $H_x \neq \varnothing \neq H^x$ for each $x \in E_0$:*

(i) *x is (H, S)-affine,*

(ii) $x \in \overline{H}_S^{\mathscr{P}}$,

(iii) *for each sequence (T_n) of positive linear operators from E_0 into F converging to S on H, we have* $\lim_{n\to\infty} T_n(x) = S(x)$.

(iv) *for each positive linear operator $T: E_0 \to F$ such that $T/_H = S/_H$, we have $T(x) = S(x)$.*

Proof. It remains to show that (iv) implies (i). First, let F be order-complete and suppose that the topology of F is order-continuous. By Lemma 1, we obtain $\sup S(H^x) = S(x) = \inf S(H_x)$. Since S is a lattice homomorphism and since the toplogy of F is order-continuous it follows that

$$S(x) = \sup S(\underline{H}^x) = \lim_{y \in \underline{H}^x} S(y) \text{ and } S(x) = \inf S(\hat{H}_x) = \lim_{y \in \hat{H}_x} S(y),$$

which yields (i).

Suppose now that E is an order-complete topological vector lattice with order-continuous topology and that S is continuous. If $x \in E_0$ and I denotes the identity operator on E there is a positive linear operator $T: E_0 \to E$ such that $T = I$ on H and $T(x) = \inf H_x$ by Lemma 1. Consequently, $S \circ T = S$ on H and $S \circ T(x) = S(\inf H_x) = S(\inf \hat{H}_x) = \lim_{y \in \hat{H}_x} S(y)$. Hence, if x satisfies condition (iv), we deduce $S(x) = \lim_{y \in \hat{H}_x} S(y)$. Replacing x by $-x$ we have (i).

Remark. In many applications neither E nor F will be order-complete vector lattices. Thus, e.g., Proposition 1 does not include the results of Scheffold [11] and Bauer [1], [2], since $\mathscr{C}(X)$ (X compact or locally compact) is not order-complete in general. Clearly, one can trivialize the problem replacing the target space $\mathscr{C}(X)$ of positive linear operators by an order-complete linear lattice G that contains $\mathscr{C}(X)$. The outcome, however, is not the desired equality between $\mathscr{A}_S(H)$ and $\overline{H}_S^{\mathscr{P}}$, since there are by far more positive linear operators from $\mathscr{C}(X)$ into G but from $\mathscr{C}(X)$ into itself (cf. [13]).

Let us now attack the hard case of non-order-complete vector lattices. Before we formulate the main theorem, we need three lemmas, the first being stated without proof, since the arguments are standard for ordered topological linear spaces (see [7], [10]).

LEMMA 2. *Let W be an order-convex*[2] *subcone of the cone of all positive linear forms on a vector lattice V. Then the following are equivalent:*

(i) *W separates the points of V.*

(ii) *There is a Hausdorff locally convex locally solid topology on V such that each continuous positive linear form on V is contained in W.*

(iii) *V_+ is closed for $\sigma(V, W - W)$.*

(iv) *$V_+ = \{z \in V: g(z) \geqslant 0$ for all $g \in W\}$.*

(v) *The closure $\overline{V}_+$ of V_+ with respect to $\sigma(V, W - W)$ is a proper cone, i.e. $\overline{V}_+ \cap -\overline{V}_+ = \{0\}$.*

In our context, we are mainly interested in the following two special cases of Lemma 2:

[2] A subset X of an ordered set Y is *order-convex* iff $\{y \in Y: x \leqslant y \leqslant x'$ *for some* $x, x' \in X\} \subset X$.

(a) *W is the cone of all positive linear forms* on *V*.

(b) *V* is a *topological vector lattice* and *W* is the *cone of all continuous positive linear forms on V* (for order-convexity of *W* see e.g. [6], p. 96, 3.3.4).

LEMMA 3. *Let E be a regularly ordered*[3] *topological vector lattice and let* $x \neq 0$ *be a positive element of* E_0. *If A is a finite subset of H such that* $A \cap H_x \neq \varnothing$ and $A \cap H^x \neq \varnothing$, *then there exists a sequence* (L_n) *of positive linear operators from E into itself satisfying* $\lim_{n\to\infty} L_n(y) = y$ *for all* $y \in A$ *and for each* $n \in \mathbb{N}$ *there is a* $y \in \hat{H}_x$ *such that* $L_n(x) \geqslant y$.

Proof. Let $a := \sum_{y \in A} | y |$ and define E_a to be the M-space $\{z \in E: | z | \leqslant \lambda a$ *for some* $\lambda \in \mathbb{R}_+\}$ with order unit norm induced by the order unit a (since the positive linear forms on E separate points, E is Archimedean, hence the order unit seminorm is a norm). By Kakutani's representation theorem there is a compact space Q and a vector lattice isomorphism $z \to \tilde{z}$ from E_a onto a dense linear sublattice $\tilde{E}_a$ of $\mathscr{C}(Q)$ such that $\tilde{a} = 1$. To define L_n, fix $n \in \mathbb{N}$ and choose a finite set $\mathscr{U}_n$ of open subsets of Q such that $(U)_{U \in \mathscr{U}_n}$ is an open covering of Q and $\tilde{y}(U)$ has diameter smaller than $1/n$ for all $y \in A$ and all $U \in \mathscr{U}_n$.

If $\langle A \rangle$ is the linear subspace of E generated by A, $E_A := \{z \in E: k \leqslant z \leqslant k'$ *for some* $k, k' \in \langle A \rangle\}$ is a vector subspace of E_a containing x. Hence, we can define the upper $\langle \tilde{A} \rangle$-envelope $\hat{\tilde{x}}: Q \to \mathbb{R}$ of $\tilde{x}$ (cf. [1]), $\hat{\tilde{x}}(\xi) =$ *inf*$\{\tilde{y}(\xi): y \in \langle A \rangle, y \geqslant x\}$. Since $\hat{\tilde{x}}$ is upper semicontinuous, for each $U \in \mathscr{U}_n$ there is a point ξ_U in the compact closure $\bar{U}$ of U in Q such that $\hat{\tilde{x}}(\xi_U) = \sup \hat{\tilde{x}}(\bar{U})$. Moreover, by Lemma 1, one can find a positive linear functional μ_U on E_A satisfying $\mu_U(y) = \tilde{y}(\xi_U)$ for all $y \in A$ and $\mu_U(\tilde{x}) = \hat{\tilde{x}}(\xi_U)$. If E_A^* is the algebraic dual of E_A, μ_U is in the $\sigma(E_A^*, E_A)$-closure of the cone G of all E_A-restrictions of positive linear forms on E. Indeed, suppose not. By Mazur's theorem, there is a $z \in E_A$ such that $\mu_U(z) < 0$ and $g(z) \geqslant 0$ for all $g \in G$. But the latter inequality implies $z \geqslant 0$ by Lemma 2, since E is regularly ordered, which yields a contradiction! Hence there is a positive linear form ω_U on E for each $U \in \mathscr{U}_n$ satisfying $| \omega_U(y) - \mu_U(y)| \leqslant 1/n$ for all $y \in A$ and $| \omega_U(x) - \mu_U(x)| \leqslant 1/n$. Finally, since $\tilde{E}_a$ is dense in $\mathscr{C}(Q)$, one can find a family $(b_U)_{U \in \mathscr{U}_n}$ of non-negative elements in E_a satisfying $1 \leqslant \sum_{U \in \mathscr{U}_n} \tilde{b}_U \leqslant 1 + 1/n$ and $\tilde{b}_U = 0$ on $Q \backslash U$ (observe that $\tilde{E}_a$ is a linear sublattice of $\mathscr{C}(Q)$!). For each $z \in E$ let now $L_n(z) = \sum_{U \in \mathscr{U}_n} \omega_U(z) \cdot b_U + \rho(z) \cdot a/n$ where ρ is a positive linear form on E such that $\rho(x) > 2$, the choice of ρ being independent of n. Clearly, L_n is a positive linear operator of E into itself. To prove the assertion of the lemma, let $y \in A$. Then $(L_n(y))_{n \in \mathbb{N}}$

[3] I.e. the positive linear forms on E separate the points of E.

converges uniformly to $\tilde{y}$ on Q: Indeed, let $\xi \in Q$ be arbitrary. We then obtain

$$|(\widetilde{L_n(y)} - \tilde{y})(\xi)| = \Big| \sum_{U \in \mathscr{U}_n} \omega_U(y)\, \tilde{b}_U(\xi) + \frac{\rho(y)}{n} - \tilde{y}(\xi)\Big|$$

$$\leqslant \Big| \sum_{U \in \mathscr{U}_n} \omega_U(y)\, \tilde{b}_U(\xi) - \sum_{U \in \mathscr{U}_n} \tilde{y}(\xi)\, \tilde{b}_U(\xi)\Big|$$

$$+ \Big| \sum_{U \in \mathscr{U}_n} \tilde{y}(\xi)\, b_U(\xi) - \tilde{y}(\xi)\Big| + \frac{|\rho(y)|}{n}$$

$$\leqslant \sum_{U \in \mathscr{U}_n'} |\omega_U(y) - \tilde{y}(\xi)|\, \tilde{b}_U(\xi) + \frac{|\tilde{y}(\xi)|}{n} + \frac{|\rho(y)|}{n}$$

where $\mathscr{U}_n' := \{U \in \mathscr{U}_n\colon \xi \in U\} \supset \{U \in \mathscr{U}_n\colon \tilde{b}_U(\xi) \neq O\}$. If $\|\tilde{y}\|$ is the uniform norm of $\tilde{y}$, we conclude:

$$|(\widetilde{L_n(y)} - \tilde{y})(\xi)| \leqslant \sum_{U \in \mathscr{U}_n'} (|\omega_U(y) - \mu_U(y)| + |\mu_U(y) - \tilde{y}(\xi)|) \cdot \tilde{b}_U(\xi)$$

$$+ \frac{\|\tilde{y}\|}{n} + \frac{|\rho(y)|}{n}$$

$$\leqslant \frac{2}{n}\Big(1 + \frac{1}{n}\Big) + \frac{\|\tilde{y}\|}{n} + \frac{|\rho(y)|}{n}.$$

Hence $\lim_{n\to\infty} \widetilde{L_n(y)} = \tilde{y}$ uniformly.

Furthermore, the inverse mapping φ of $z \to \tilde{z}$ is continuous, since the image of the unit ball in $\tilde{E}_a$ is the order interval $[-a, a]$ which is absorbed by each (solid) zero-neighborhood in F. Consequently, $\lim_{n\to\infty} L_n(y) = y$.

Finally, let $n \in \mathbb{N}$ and $\xi \in Q$ be arbitrary. Then the following inequalities hold:

$$\widetilde{L_n(x)}(\xi) > \sum_{U \in \mathscr{U}_n} \omega_U(x)\, \tilde{b}_U(\xi) + \frac{2}{n} \geqslant \sum_{U \in \mathscr{U}_n} \Big(\mu_U(x) - \frac{1}{n}\Big)\, \tilde{b}_U(\xi) + \frac{2}{n}$$

$$\geqslant \sum_{U \in \mathscr{U}_n} \hat{\tilde{x}}(\xi_U)\, \tilde{b}_U(\xi) - \frac{1}{n}\Big(1 + \frac{1}{n}\Big) + \frac{2}{n}$$

$$= \sum_{U \in \mathscr{U}_n'} (\sup \hat{\tilde{x}}(\overline{U}))\, \tilde{b}_U(\xi) - \frac{1}{n}\Big(1 + \frac{1}{n}\Big) + \frac{2}{n} \quad (\mathscr{U}_n' \text{ defined as above})$$

$$\geqslant \hat{\tilde{x}}(\xi) \sum_{U \in \mathscr{U}_n'} \tilde{b}_U(\xi) \geqslant \hat{\tilde{x}}(\xi).$$

Hence, for each $\xi \in Q$ there is a function $\tilde{y}_\xi \in \langle \tilde{A} \rangle$ majorizing $\tilde{x}$ and such that $\tilde{y}_\xi(\xi) < \widetilde{L_n(x)}(\xi)$. This inequality is still valid in a whole neighborhood of ξ. Since a finite number of such neighborhoods covers Q, there exists a finite subset $Y \subset \{y \in \langle A \rangle: y \geqslant x\}$ such that $\inf(\tilde{Y}) < \widetilde{L_n(x)}$ on Q. But $Y = \varphi(\tilde{Y}) \subset H_x$ and $\varphi(\inf \tilde{Y}) = \inf \varphi(\tilde{Y}) \in \hat{H}_x$. This completes the proof.

LEMMA 4. *Let E be a regularly ordered topological vector lattice, and let $x \in E_+ \backslash \{0\}$ be such that $y \leqslant x \leqslant y'$ for some $y, y' \in H$. Then there is a net $(L_i)_{i \in I}$ of positive linear operators from E into itself satisfying* $\lim_{i \in I} L_i(y) = y$ *for all $y \in H$ and $\{L_i(x): i \in I\} \subset \hat{H}_x + E_+$. If the continuous positive linear forms on E separate the points of E, then the operators L_i can be choosen continuous. Moreover, if H has a countable algebraic basis and the topology of E is first countable, the net (L_i) can be replaced by a sequence of positive linear operators (resp. of continuous positive linear operators) on E.*

Proof. Let $\mathfrak{H} := \{A \subset H: A$ *finite*, $A \cap H_x \neq \varnothing$ *and* $A \cap H^x \neq \varnothing\}$. By Lemma 3, for each $A \in \mathfrak{H}$ there is a sequence $(L_n^A)_{n \in \mathbb{N}}$ of positive endomorphisms on E satisfying

$$\lim_{n \to \infty} L_n^A(y) = y$$

whenever $y \in A$ and

$$\{L_n(x): n \in \mathbb{N}\} \subset \hat{H}_x + E_+ .$$

Let $I := \mathbb{N}^{\mathfrak{H}} \times \mathfrak{H}$. We define an ordering on I setting $((n_A)_{A \in \mathfrak{H}}, B) \leqslant ((n'_A)_{A \in \mathfrak{H}}, B')$ iff $n_A \leqslant n'_A$ (for all $A \in \mathfrak{H}$) and $B \subset B'$. Obviously, I is upward directed. For $i = ((n_A)_{A \in \mathfrak{H}}, B) \in I$ let $L_i := L^B_{n_B}$.

Since $\{L_i(x): i \in I\} \subset \hat{H}_x + E_+$ clearly follows from the corresponding inclusion for the sequences (L_n^A), it remains to prove the first assertion: Let $y \in H$ and let V be a zero-neighborhood in E. For each $A \in \mathfrak{H}$ containing y choose a natural number m_A such that $L_n^A(y) - y \in V$ for all $n \geqslant m_A$. If $A \in \mathfrak{H}$ does not contain y, define $m_A := 1$. Select a set $A_0 \in \mathfrak{H}$ satisfying $y \in A_0$. Then $L_i(y) - y \in V$ whenever $i \geqslant ((m_A)_{A \in \mathfrak{H}}, A_0)$ $(i \in I)$. Indeed, let $i = ((n_A)_{A \in \mathfrak{H}}, B) \geqslant ((m_A)_{A \in \mathfrak{H}}, A_0)$. Since $y \in A_0 \subset B$ and $n_B \geqslant m_B$, it follows that $L_i(y) - y = L^B_{n_B}(y) - y \in V$.

Suppose now that the continuous positive linear forms on E separate the points of E; then it is clear from the proof of Lemma 3 that the operators L_n^A can be choosen continuous. Hence the net (L_i) of positive endomorphisms on E defined above consists of continuous operators.

Finally, suppose that $(y_n)_{n \in \mathbb{N}}$ is a basis of H. By Lemma 3, for each $m \in \mathbb{N}$ there is a sequence $(L_n^m)_{n \in \mathbb{N}}$ of positive endomorphisms on E satisfying

$$\lim_{n \to \infty} L_n^m(y_j) = y_j \text{ for all } j = 1, \ldots, m \text{ and } \{L_n^m(x) : n \in \mathbb{N}\} \subset \hat{H}_x + E_+$$

Let d be a pseudo-metric on E generating the topology of E. For each $n \in \mathbb{N}$ there is a natural number k_n such that

$$d(L_{k_n}^n(y_i), y_i) < \frac{1}{n} \text{ for all } i = 1,\ldots, n.$$

If we set $L_n := L_{k_n}^n (n \in \mathbb{N})$, then obviously $\lim_{n\to\infty} L_n(y_i) = y_i$ for all $i \in \mathbb{N}$. Hence $\lim_{n\to\infty} L_n(y) = y$ for all $y \in H$. $\{L_n(x): n \in \mathbb{N}\} \subset \hat{H}_x + E_+$ again follows from the corresponding inclusion for the operators L_n^m.

Lemma 4 now yields the following converse of Theorem 1:

THEOREM 2. *Let E be regularly ordered. Then the following statements are equivalent for $x \in E_0$:*

(i) x *is (H, S)-affine*

(ii) $x \in \overline{H}_S^{\mathscr{P}}$.

Moreover, if E is a locally convex vector lattice and S is continuous, then the list of equivalent statements can be extended by

(iii) $x \in \overline{H}_S^{\mathscr{P}_c}$, *where $\mathscr{P}_c$ denotes the class of all nets of continuous positive linear operators from E_0 into F.*

Proof. By Theorem 1 we have only to prove the implications (ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (i). If E is endowed with the initial topology with respect to S, then E will become a topological vector lattice such that S is continuous. The rest of the proof is carried out in Theorem 1.2 of [5].

Let $\mathscr{P}'$, $\mathscr{P}_c'$ denote the subset of all sequences in $\mathscr{P}$, $\mathscr{P}_c$, respectively. An example of Scheffold [11] shows that the inclusions $\overline{H}_S^{\mathscr{P}} \subset \overline{H}_S^{\mathscr{P}'}$, $\overline{H}_{S}^{\mathscr{P}_c'}$ cannot be replaced by equalities in general. Using an obvious modification of a proof presented in [5] based on Lemma 4, we have equality provided that the topology of F is first countable, H has a countable basis and E is regularly ordered (respectively E is a locally convex vector lattice and S is continuous). Moreover, if $E = E_0 = \{x \in E: y \leqslant x \leqslant y' \text{ for some } y, y' \in H\}$ is a dense linear sublattice of $\mathscr{C}(X)$ endowed with the topology of locally uniform convergence, X locally compact, countable at infinity, H is separable and S is continuous, then we also have the equality

$$\mathscr{A}_S(H) = \overline{H}_S^{\mathscr{P}} = \overline{H}_S^{\mathscr{P}'}.$$

To prepare the proof we need the following lemma

LEMMA 5. *Let X be a locally compact σ-compact space and let E_0 be a dense vector sublattice of $\mathscr{C}(X)$ endowed with the topology of uniform convergence on all compact subsets of X. Furthermore, suppose that H is separable* *and* $H_f \neq \varnothing \neq H^f$ *for each* $f \in E_0$. *Then, given a positive function* $f \in E_0$, *there exists a sequence* (L_n) *of positive linear operators of* E_0 *into itself satisfying*

(i) $\lim_{n\to\infty} L_n(h) = h$ *for all* $h \in H$,

(ii) *for each compact subset* $K \subset X$ *there is a natural number* n_0 *such that* $\inf_{n\geqslant n_0} L_n(f)(y) \geqslant \hat{f}(y) := \inf_{h\in H_f} h(y)$ *for all* $y \in K$,

(iii) *if all positive linear forms on* E_0 *are continuous, then each* L_n *is continuous.*

Proof. Since E_0 is a linear lattice, we can assume that $E = E_0$. Choose a sequence (K_n) of compact subsets of X such that $K_n \subset \mathring{K}_{n+1}$ (= interior of K_{n+1}) for all $n \in \mathbb{N}$ and $\bigcup_{n\in\mathbb{N}} K_n = X$. Let $n \in \mathbb{N}$ and $x \in X$ be arbitrary. Then there is a (uniquely determined) natural number m such that $x \in K_m \backslash K_{m-1}$ (define $K_0 := \varnothing$). Hence the set

$$U_x^n := \{y \in \mathring{K}_{m+1}\backslash K_{m-1}: |h_j(y) - h_j(x)| < 1/2n \quad \text{for all} \quad j \leqslant n \quad (j \in \mathbb{N})\}$$

where (h_j) is a dense sequence in H, is an open relatively compact neighborhood of x.

Since $(U_x^n)_{x\in K_n}$ is an open covering of K_n, there is a finite subset $S_n \subset K_n$ such that $\bigcup_{x\in S_n} U_x^n \supset K_n$. Moreover, we can find a family $(\sigma_x^n)_{x\in S_n}$ of positive functions in $E = E_0$ with the following properties (observe that E_0 is a dense sublattice of $\mathscr{C}(X)$!):

$$1 \leqslant \sum_{x\in S_n} \sigma_x^n(y) \leqslant 1 + \frac{1}{n} \text{ for all } y \in K_n,$$

$$\sigma_x^n(y) = 0 \text{ for all } y \in K_n \backslash U_x^n \ (x \in S_n).$$

For each $x \in S_n$ choose a point $y_x^n \in \overline{U_x^n}$ and an H-representing (cf. 1) functional ω_x^n for y_x^n such that $\hat{f}(y_x^n) = \sup \hat{f}(\overline{U_x^n})$ and $\omega_x^n(f) = \hat{f}(y_x^n)$. Let $L_n: E \to E$ be given by $L_n(g) = \sum_{x\in S_n} \omega_x^n(g)\, \sigma_x^n$ $(g \in E)$. Since L_n is evidently continuous, if all positive linear forms on E are continuous, it remains to show (i) and (ii): Thus, let $h \in H$ and consider a compact subset K of X. Define $\alpha := \sup_{y\in K} |h(y)|$ and, for each $\epsilon > 0$ $(\epsilon \in \mathbb{R})$, choose $n_0, n_1 \in \mathbb{N}$ with the following properties:

$K_{n_0} \supset K$ (this is possible, since $(\mathring{K}_n)$ covers K and (K_n) is increasing), $n_0\epsilon \geqslant 4 + \alpha$,

$$|h_{n_1}(y) - h(y)| \leqslant \frac{1}{2n_0} \text{ for all } y \in K_{n_0+1}.$$

Then for each natural number $n \geqslant max(n_0, n_1)$ and each $y \in K$ we obtain:

$$|(L_n(h) - h)(y)| = \sum_{x \in S_n} \omega_x^n(h)\, \sigma_x^n(y) - h(y)$$

$$\leqslant \sum_{x \in S_n} |h(y_x^n) - h(y)|\, \sigma_x^n(y) + \left| \sum_{x \in S_n} \sigma_x^n(y) - 1 \right| |h(y)|$$

$$\leqslant \sum_{x \in S_n} |h(y_x^n) - h(y)|\, \sigma_x^n(y) + \frac{\alpha}{n}.$$

Furthermore, the following estimate holds for each $x \in S_n$:

$$|h(y_x^n) - h(y)| \leqslant |h(y_x^n) - h_{n_1}(y_x^n)| + |h_{n_1}(y_x^n) - h_{n_1}(y)|$$

$$+ |h_{n_1}(y) - h(y)|.$$

If $x \notin K_{n_0}$, then $U_x^n \cap K_{n_0} = \varnothing$, consequently $\sigma_x^n(y) = 0$. Thus, for $\sigma_x^n(y) \neq 0$, we must have $x \in K_{n_0}$ and $y_x^n \in \overline{U_x^n} \subset K_{n_0+1}$. From this we conclude: $\sigma_x^n(y)|\, h(y_x^n) - h(y)| \leqslant (1/n_0 + (|h_{n_1}(y_x^n) - h_{n_1}(y)|)\, \sigma_x^n(y) \leqslant (1/n_0 + |h_{n_1}(y_x^n) - h_{n_1}(x)| + |h_{n_1}(x) - h_{n_1}(y)|)\, \sigma_x^n(y) \leqslant 2/n_0\, \sigma_n^x(y)$ by the definition of U_x^n. But this yields:

$$|(L_n(h) - h)(y)| \leqslant \frac{2}{n_0} \cdot \left(1 + \frac{1}{n}\right) + \frac{\alpha}{n} \leqslant \frac{4 + \alpha}{n_0} \leqslant \epsilon \text{ which proves (i).}$$

(ii) Let K again be an arbitrary compact subset of X and choose $n_0 \in \mathbb{N}$ such that $K_{n_0} \supset K$. Then for each $n \geqslant n_0$ $(n \in \mathbb{N})$ and each $y \in K$ we obtain: $L_n(\hat{f})(y) = \sum_{x \in S_n} \omega_x^n(f)\, \sigma_x^n(y) = \sum_{x \in S_n} \hat{f}(y_x^n)\, \sigma_x^n(y) = \sum_{x \in S_n'} \hat{f}(y_x^n)\, \sigma_x^n(y)$ where $S_n' := \{x \in S_n : y \in U_x^n\}$, hence $L_n(f)(y) \geqslant \sum_{x \in S_n'} \hat{f}(y)\, \sigma_x^n(y) \geqslant \hat{f}(y)$, since $\hat{f}(y_x^u) = \sup \hat{f}(\overline{U_x^n})$. This completes the proof.

THEOREM 3. *In addition to the assumptions of Lemma 5 suppose that the vector lattice homomorphism $S\colon E_0 \to F$ is continuous. Then the following are equivalent for each function $f \in E_0$:*

(i) *f is (H, S)-affine.*

(ii) $f \in \overline{H}_S^{\mathscr{P}'}$.

Moreover, if all positive linear forms on E_0 are continuous, we can adjoin the further equivalent condition

(iii) $f \in \overline{H}_S^{\mathscr{P}'_c}$.

Proof. Following the proof of Theorem 2.1 in [5] it suffices to show that, for each positive function $f \in E_0$, (ii) implies $\lim_{g \in \hat{H}_f} S(g) = S(f)$. Thus, let $f \in E_0$ be positive and such that (ii) holds. If (L_n) is a sequence of positive linear operators on E_0 satisfying the conditions (i), (ii) of Lemma 5, we define the sequence (T_n) of positive linear operators from E_0 into F by $T_n = S \circ L_n$ $(n \in \mathbb{N})$.

The continuity of S yields $\lim_{n\to\infty} T_n(h) = S(h)$ for all $h \in H$. Hence it follows from the assumption on f that $\lim_{n\to\infty} T_n(f) = S(f)$. Let $\mathscr{U}_F(0)$ be the system of zero-neighborhoods in F. The proof will be complete, if we can show that for each solid $U \in \mathscr{U}_F(0)$ there is a function $g \in \hat{H}_f$ such that $S(g) - S(f) \in U$. Indeed, this implies $\lim_{g \in \hat{H}_f} S(g) = S(f)$, since $\hat{H}_f$ is downward directed and U is solid.

Thus, let $U \in \mathscr{U}_F(0)$ be solid and choose $U' \in \mathscr{U}_F(0)$ such that $U' + U' \subset U$. Since $\lim_{n\to\infty} T_n(f) = S(f)$, there is an $n_0 \in \mathbb{N}$ satisfying $T_n(f) - S(f) \in U'$ for all $n \geqslant n_0$ $(n \in \mathbb{N})$. Moreover, from the continuity of S we derive the existence of a compact $K \subset X$ and a positive real number δ such that $|g| < \delta$ on K implies $S(g) \in U'$ whenever $g \in E$. By Lemma 5 $\inf_{n \geqslant n_1} L_n(f)(y) \geqslant \hat{f}(y)$ on K for some $n_1 \in \mathbb{N}$. Since $\inf \hat{H}_f = \hat{f}$, it follows that $\inf_{g \in \hat{H}_f} (g - L_n(f))^+(y) = 0$ for all $y \in K$ and all $n \geqslant n_1$ $(n \in \mathbb{N})$. By Dini's theorem and the compactness of K there exists a function $g \in \hat{H}_f$ satisfying $|(g - L_j(f))^+(y)| < \delta$ for all $y \in K$ where $j := \max(n_0, n_1)$. Hence $S((g - L_j(f))^+) \in U'$ and consequently

$$\begin{aligned} S(f) &\leqslant S(g) \leqslant S(L_j(f)) + S((g - L_j(f))^+) \\ &= T_j(f) + S((g - L_j(f))^+) \in S(f) + U' + U' \in S(f) + U. \end{aligned}$$

References

1. H. Bauer, Theorems of Korovkin type for adapted spaces, *Ann. Inst. Fourier (Grenoble)* **23**, No. 4 (1973), 245–260.
2. H. Bauer, Convergence of monotone operators, *Math. Z.* **136** (1974), 315–330.
3. H. Berens and G. G. Lorentz, Theorems of Korovkin type for positive linear operators on Banach lattices, *in* "Approximation Theory" (G. G. Lorentz, Ed.), Academic Press, New York/London, 1973.
4. H. Berens and G. G. Lorentz, Geometric theory of Korovkin sets, *J. Approximation Theory* **15** (1975), 161–189.
5. K. Donner, Korovkin theorems for positive linear operators, *J. Approximation Theory* **13** (1975), 443–450.
6. G. Jameson, "Ordered Linear Spaces," Lecture Notes in Mathematics No. 141, Springer–Verlag, Berlin/Heidelberg/New York, 1970.
7. A. L. Pressini, "Ordered Topological Vector Spaces," Harper & Row, New York, 1967.
8. Yu. A. Saskin, Korovkin systems in spaces of continuous functions, *Amer. Math. Soc. Transl.* (2) **54** (1966), 125–144.
9. Yu. A. Saskin, The Milman–Choquet boundary and approximation theory, *Functional Anal. Appl.* **1** (1967), 170–171.

10. H. H. SCHAEFER, "Topological Vector Spaces," Springer–Verlag, Berlin/Heidelberg/ New York, 1971.
11. E. SCHEFFOLD, Über die punktweise Konvergenz von Operatoren $C(X)$, *Rev. Acad. Ci. Zaragoza* **28** (1973), 5–12.
12. E. SCHEFFOLD, Ein allgemeiner Korovkin-Satz für lokalkonvexe Vektorverbände, *Math. Z.* **132** (1973), 209–214.
13. R. K. VASIL'EV, The conditions for the convergence of isotonic operators in partially ordered sets with convergence classes, *Math. Notes* **12** (1972), 632–637.
14. M. WOLFF, Über Korovkin-Sätze in lokalkonvexen Vektorverbänden, *Math. Ann.* **204** (1973), 49–56.
15. M. WOLFF, On Korovkin-type theorems in special function lattices, *J. Approximation Theory* **22** (1978), 243–253.
16. D. E. WULBERT, Convergence of operators and Korovkin's theorem, *J. Approximation Theory* **1** (1968), 381–390.